1/30/04 ER 11438

CS 3/12/04

Document And Report Documentation Page Submitted as edoc_1075488490

Report Documentation Page	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
12 MAR 2003	**N/A**	**-**

4. TITLE AND SUBTITLE **Mean Squared Error Performance Prediction of Maximum-Likelihood signal Parameter Estimation**	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Defense Advanced Research Project Agency (DARPA)**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
Also see: ADM001520 , The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:	17.	18.	19a. NAME OF RESPONSIBLE

a. REPORT	b. ABSTRACT	c. THIS PAGE	LIMITATION OF ABSTRACT	NUMBER OF PAGES	PERSON
unclassified	**unclassified**	**unclassified**	**UU**	**21**	**Patricia Mawby, EM 1438** **PHONE:(703) 767-9038** **EMAIL:pmawby@dtic.mil**

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

pwd: cannot determine current directory!

Mean Squared Error Performance Prediction of Maximum-Likelihood Signal Parameter Estimation

Christ D. Richmond

Adaptive Senor Array Processing Workshop
Session III: Adaptive Detection and Estimation

12th March 2003

*This work was sponsored by DARPA under Air Force contract F19628-95-C-0002. Opinions, interpretations, conclusions, and recommendations are those of the author and are not necessarily endorsed by the United States Government.

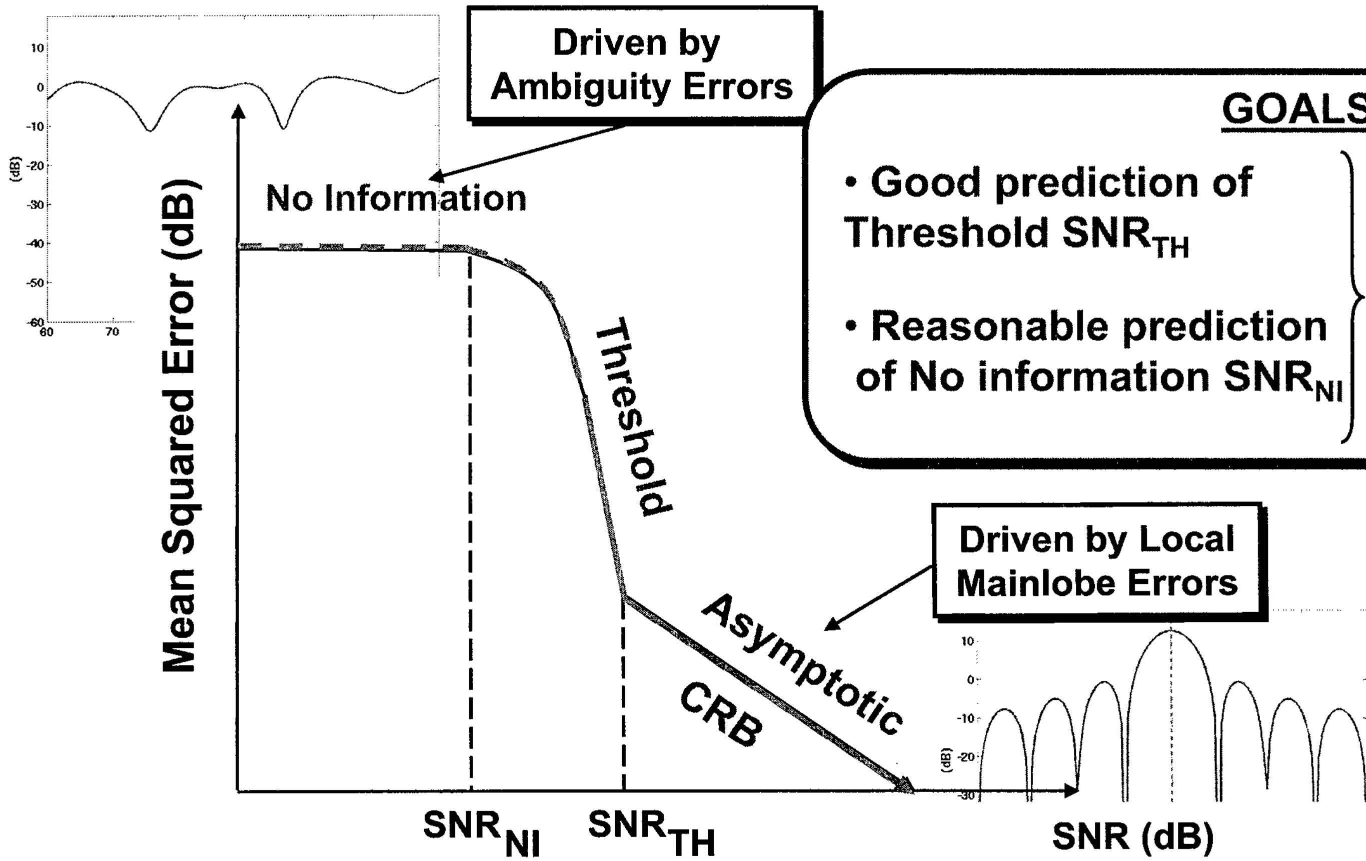
Outline

- Introduction
  - Problem
  - Previous Work
- Theory
- Numerical Results
- Conclusions

Goals of Analysis

- **Problem:**
  - Mean Squared Error (MSE) performance of Maximum-Likelihood (ML) signal parameter estimation unknown
    1. Colored Noise
    2. Finite Number of Colored Noise Only Training Samples
- **Goal:**
  - Develop robust theory for prediction of ML MSE
- **Proposed Method:**
  - Use Interval Error based method proposed by Van Trees 1968
    - Must derive/approximate probability of “interval error”

Typical Composite MSE Performance



GOALS:

- Good prediction of Threshold SNR_{TH}
- Reasonable prediction of No information SNR_{NI}

- Colored Noise
- Adaptive Training

- Three definitive regions of Signal-to-Noise-Ratio (SNR)
  - No Information, Threshold, and Asymptotic (CRB)
- Recall MSE = Estimator Variance + Estimator Bias

Previous Work

- K. Bell, Ph. D. George Mason University, 1995
- K. Bell, Y. Steinberg, Y. Ephraim, H. Van Trees, IEEE T-SP March 1997
- S. Pawlukiew... 1999
- F. Athley, ...

What's New ?

- Colored Noise Allowed
- Colored Noise Only Finite training effects
- Exact two point error probabilities used

Ziv-Zakai Bounds

Method of Interval Error

- All previous work considered non-adaptive and white noise only case
- Error probabilities approximated via Chernoff Bounds

Outline

- Introduction
- Theory
  - Maximum-Likelihood Estimation (MLE)
  - Interval Error Based Method of MSE Prediction
- Numerical Results
- Conclusions

Maximum-Likelihood Signal Parameter Estimation

Data Model: $\pi^{-N}|\mathbf{R}|^{-1}\exp\left\{-[\mathbf{x}-S\mathbf{v}(\theta)]^H\mathbf{R}^{-1}[\mathbf{x}-S\mathbf{v}(\theta)]\right\}$ $\boxed{S \text{ unknown}}$

ML Estimator: $\theta_{ML} = \arg\max t_{MF}(\theta)$ $\quad t_{MF}(\theta)=\dfrac{\left|\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{x}\right|^2}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$ ***Clairvoyant Matched Filter***

Data Model: $\pi^{-N(L+1)}|\mathbf{R}|^{-(L+1)}\exp\left\{-[\mathbf{x}-S\mathbf{v}(\theta)]^H\mathbf{R}^{-1}[\mathbf{x}-S\mathbf{v}(\theta)]-\mathrm{tr}\left(\mathbf{R}^{-1}\mathbf{X}\mathbf{X}^H\right)\right\}$ $\boxed{\mathbf{R} \text{ unknown} \quad S \text{ unknown}}$

ML Estimator: $\theta_{ML} = \arg\max t_{AMF}(\theta)$ $\quad t_{AMF}(\theta)=\dfrac{\left|\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{x}\right|^2}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$ $\quad \hat{\mathbf{R}} \equiv \dfrac{1}{L}\mathbf{X}\mathbf{X}^H$ ***Adaptive Matched Filter***

- **Complex Gaussian data model: All snapshots N x 1**
  - **Arbitrary N x N Colored Covariance**
  - **Deterministic Signal ("Conditional")**
- **Colored noise only training samples available**
- **Single scalar signal parameter**
  - **Joint signal parameter estimation not considered**

Method of ML MSE Prediction: Based on Interval Errors

- In general MSE can be written as the sum of two terms

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \Pr(\text{No Interval Error})\,E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{No Interval Error}\right\}$$
$$+\Pr(\text{Interval Error})\,E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{Interval Error}\right\}$$

- MSE for Deterministic Signal Parameters

SINR Loss!!!

$$\boxed{E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \theta_k\right\} \cong \left[1-\sum_{\substack{n=1\\n\neq k}}^{K} p\left(\hat{\theta}=\theta_n \middle| \theta_k\right)\right] \mathrm{CRB}(\theta_k) + \sum_{\substack{n=1\\n\neq k}}^{K} p\left(\hat{\theta}=\theta_n \middle| \theta_k\right)\left(\theta_n-\theta_k\right)^2}$$

- Challenge is accurate calculation of error probabilities given by

$$p\left(\hat{\theta}=\theta_n \middle| \theta_k\right) = ?$$

Union Bound (UB) Approximation: Interval Error Probabilities

- **Recall the ML approach:** $\hat{\theta} = \underset{\theta_1, \theta_2, \ldots, \theta_K}{\arg\max}\, t(\theta)$

- **The probability of interval error is bounded by the relation**

UNION BOUND

$$p\left(\hat{\theta} = \theta_n \middle| \theta_k\right) = \Pr\left\{ \bigcup_{\substack{k=1 \\ k \neq n}}^{K} \left[t(\theta_n) > t(\theta_k) \middle| \theta = \theta_k \right] \right\} \leq \sum_{\substack{k=1 \\ k \neq n}}^{K} \Pr\left[t(\theta_n) > t(\theta_k) \middle| \theta = \theta_k \right]$$

- **UB is a useful tool for computation of error probabilities in Digital Communication Schemes**
  - **Approximation relies on <u>two point error probabilities</u>**
- **UB often over estimates error in "No Information" region of the MSE curve**

Two Point Probabilities for the Matched Filter: R known

- Let array responses for two points be given by $\mathbf{V} = \left[\mathbf{v}(\theta_n), \mathbf{v}(\theta_k)\right]$
- Define the following matrices

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{\mathbf{v}(\theta_n)\mathbf{R}^{-1}\mathbf{v}(\theta_n)}} & 0 \\ 0 & \frac{1}{\sqrt{\mathbf{v}(\theta_k)\mathbf{R}^{-1}\mathbf{v}(\theta_k)}} \end{bmatrix} \qquad \mathbf{W} = \mathbf{R}^{-1}\mathbf{V}\mathbf{A} \qquad \mathbf{R}_{VX} \equiv \mathbf{A}\mathbf{V}^H\mathbf{R}^{-1}\mathbf{V}\mathbf{A}$$

$$\mathbf{R}_{VX}^{1/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{R}_{VX}^{1/2} = \mathbf{Q}_{VX}^H \Lambda_{VX} \mathbf{Q}_{VX}$$

- Defining the vector $\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \mathbf{Q}_{VX}\mathbf{R}_{VX}^{-1/2}\mathbf{W}^H\mathbf{v}(\theta_k)$

The exact desired two point probabilities are given by

$$\boxed{\Pr\left[t_{MF}(\theta_n) > t_{MF}(\theta_k) \middle| \theta = \theta_k\right] = \Pr\left[\frac{\chi_1^2\left(|m_1|^2\right)}{\chi_1^2\left(|m_2|^2\right)} \le \frac{-\lambda_{VX,2}}{\lambda_{VX,1}}\right]}$$

- Expressible in terms of Marcum Q-function

Two Point Probabilities for the Adaptive Matched Filter: R unknown

- Let $\begin{bmatrix} t_{AMF}(\theta_n) \\ t_{AMF}(\theta_k) \end{bmatrix} = \begin{bmatrix} |y_{AMF,1}|^2 \\ |y_{AMF,2}|^2 \end{bmatrix}$; the desired probability can be written

$$\Pr\left[t_{AMF}(\theta_n) > t_{AMF}(\theta_k) \middle| \theta = \theta_k \right] = \Pr\left\{ \mathbf{y}_{AMF}^H \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}_{AMF} < 0 \right\}$$

- It can be shown that AMF outputs can be written equal in distribution

$$\mathbf{y}_{AMF} = \begin{bmatrix} y_{AMF,1} \\ y_{AMF,2} \end{bmatrix} \stackrel{d}{=} \begin{bmatrix} \sqrt{a_{11}} & \frac{a_{12}}{\sqrt{a_{11}}} \\ \frac{a_{21}}{\sqrt{a_{22}}} & \sqrt{a_{22}} \end{bmatrix} \left(\mathbf{V}^H \mathbf{R}^{-1} \mathbf{V} \right)^{-1/2} \mathbf{x}_{AMF} \quad \text{where}$$

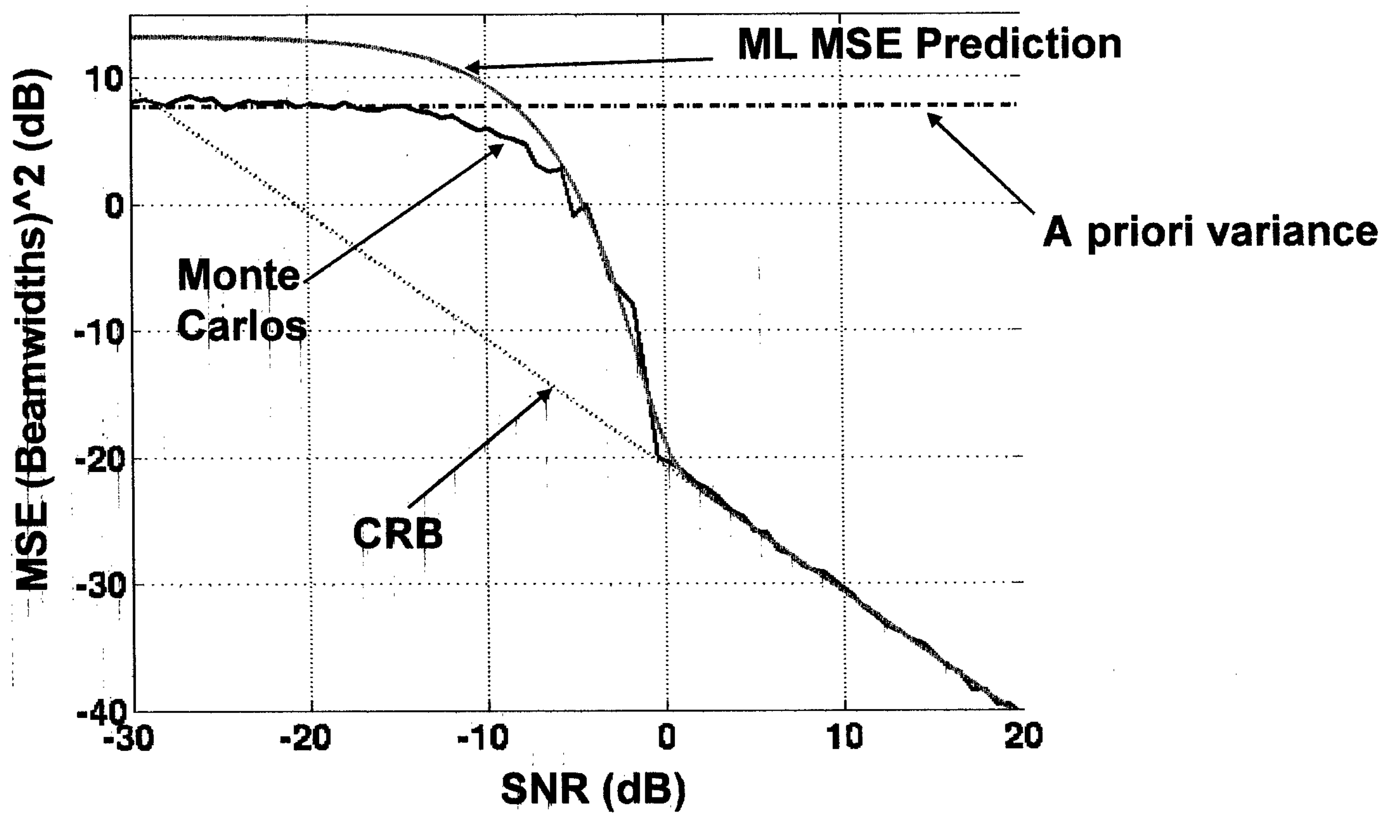
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim CW\left(L - N + 2, \mathbf{V}^H \mathbf{R}^{-1} \mathbf{V} \right) \text{ and } \mathbf{x}_{AMF} \sim CN_{2\text{x}1}\left(S \begin{bmatrix} \sqrt{\mathbf{v}(\theta_n) \mathbf{R}^{-1} \mathbf{v}(\theta_n)} \\ 0 \end{bmatrix}, \mathbf{I}_2 \cdot \frac{1}{\beta_{L-N+3,N-2}} \right)$$

- The necessary two point probabilities can be thus obtained

Outline

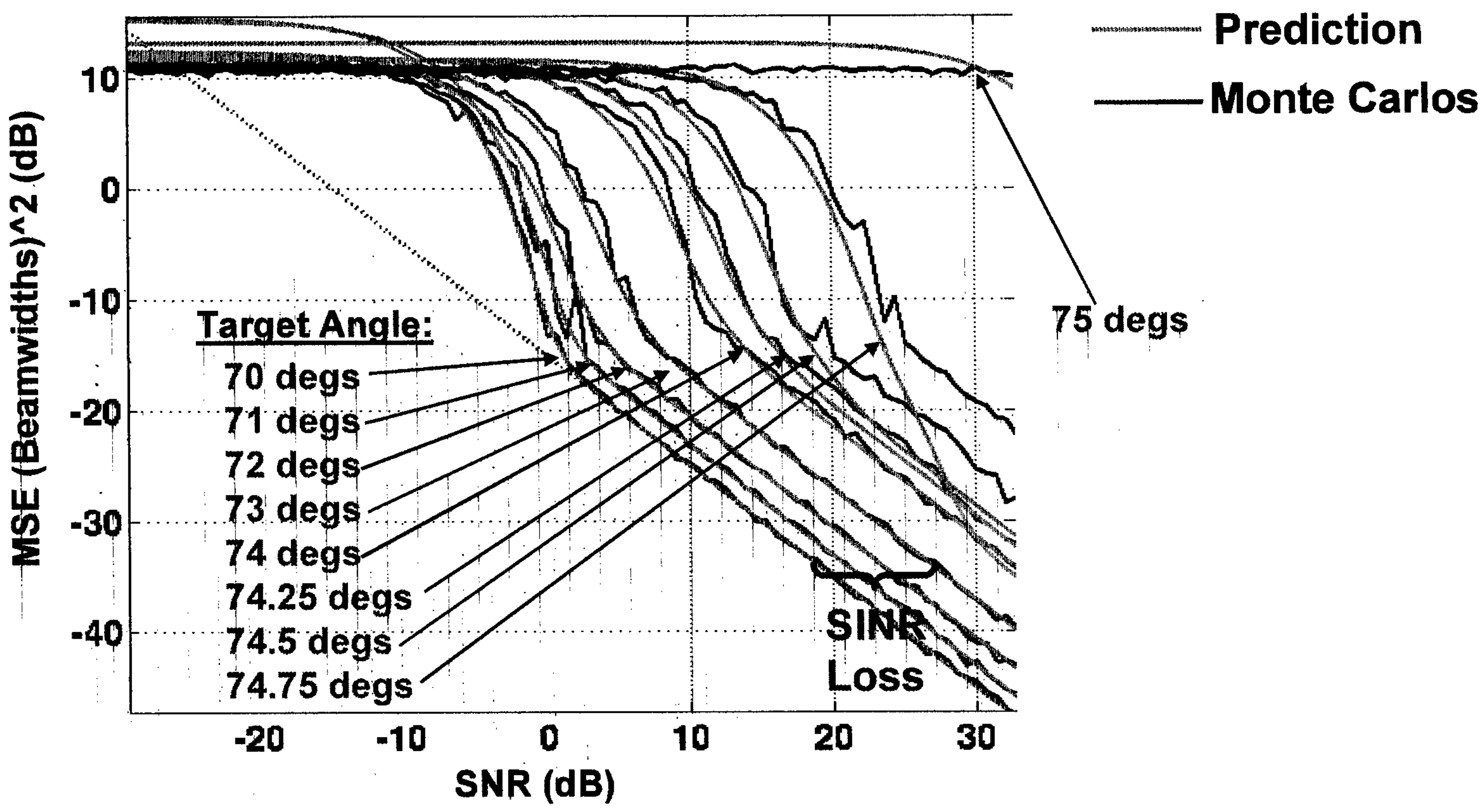
- Introduction
- Theory
- Numerical Results
- Conclusions

White Noise Example: R known



- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise

Colored Noise Example: R known



- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise plus 30dB Jammer at 75 degs

White Noise Example: R unknown

Need Figure

- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise
- Adaptive Training: L = 1.5N, 2N, and 3N

Colored Noise Example:
R unknown

Need Figure

- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise plus 30dB Jammer at 75 degs
- Adaptive Training: L = 1.5N, 2N, and 3N

Conclusions

- Interval error method represents a viable and numerically efficient technique
  - Theory and simulation have very good match
  - UB overestimates MSE, however, in “No Information” region
- Two point probabilities have been computed in closed form
  - Colored Noise
  - Adaptive Finite Training Effects
- Established a the notion of SINR Loss for the parameter estimation problem

Future Work

- Explore tighter bounds on probability of interval errors than that given by the Union Bound
  - Expurgating terms of Union Bound, for example
- Extend to Stochastic / Unconditional signal models
- Generalize to vector signal parameters
- Comparisons with Bayesian Bound predictions
  - Ziv-Zakai, Weiss-Weinstein, etc.

Backups

Method of ML MSE Prediction: Based on Interval Errors

- **In general MSE can be written as the sum of two terms**

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \Pr(\text{No Interval Error})\,E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{No Interval Error}\right\}$$
$$+\Pr(\text{Interval Error})\,E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{Interval Error}\right\}$$

- **Deterministic Signal Parameters**

$$E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \theta_k\right\} = \Pr\left(\text{No Interval Error}\middle|\theta_k\right)\cdot \text{CRB}(\theta_k) + \sum_{\substack{n=1\\ n\neq k}}^{K} p\left(\hat{\theta}=\theta_n \,\middle|\, \theta_k\right)\left(\theta_n-\theta_k\right)^2$$

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \int_{\hat{\Theta}}\left(\hat{\theta}-\theta\right)^2 p\left(\hat{\theta}\,|\,\theta\right)d\hat{\theta}$$

$$E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \theta\right\} = \int_{\hat{\Theta}:MAINLOBE}\left(\hat{\theta}-\theta\right)^2 p\left(\hat{\theta}\,|\,\theta\right)d\hat{\theta} + \int_{\hat{\Theta}:AMBIGUITIES}\left(\hat{\theta}-\theta\right)^2 p\left(\hat{\theta}\,|\,\theta\right)d\hat{\theta}$$

Two Point Probabilities for the Matched Filter: R known

- These probabilities are expressible in terms of the Marcum Q-function:

$$\Pr\left[t_{MF}(\theta_n) > t_{MF}(\theta_k) \mid \theta = \theta_k\right] = \Pr\left[\frac{\chi_1^2\left(|m_1|^2\right)}{\chi_1^2\left(|m_2|^2\right)} \le \frac{-\lambda_{VX,2}}{\lambda_{VX,1}}\right] =$$

$$\left[\frac{\lambda_{VX,2}}{\lambda_{VX,2} - \lambda_{VX,1}}\right] \cdot \left\{ \frac{-\lambda_{VX,1}}{\lambda_{VX,2}} Q_1\left(\sqrt{\frac{2|m_1|^2 \lambda_{VX,1}}{\lambda_{VX,1} - \lambda_{VX,2}}}, \sqrt{\frac{2|m_2|^2 \lambda_{VX,2}}{\lambda_{VX,2} - \lambda_{VX,1}}}\right) + \left[1 - Q_1\left(\sqrt{\frac{2|m_2|^2 \lambda_{VX,2}}{\lambda_{VX,2} - \lambda_{VX,1}}}, \sqrt{\frac{2|m_1|^2 \lambda_{VX,1}}{\lambda_{VX,1} - \lambda_{VX,2}}}\right)\right]\right\}$$

Return